

ASPECTS OF DUALITY AND CONFINING STRINGS

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We inspect the excitation energy spectrum of a confining string in terms of solitons in an effective field theory model. The spectrum can be characterized by a spectral function, and twisting and bending of the string is manifested by the invariance of this function under a duality transformation. Both general considerations and numerical simulations reveal that the spectral function can be approximated by a simple rational form, which we propose becomes exact in the Yang-Mills theory.

The ground state energy of a long confining string is proportional to its length L . When L decreases there will be corrections, and in QCD we expect for the ground state energy of the confining flux [1]

$$E_0(L) = \sigma \cdot L + \varepsilon + \frac{c}{L} + \mathcal{O}\left(\frac{1}{L^2}\right) \quad (1)$$

Here σ is the string tension, ε the intercept, and $c = \frac{\pi}{12}$ is a universal constant, the Casimir energy of zero-point fluctuations that can be computed from the Nambu-Goto action in the Gaussian approximation [1]. Unfortunately, a simple Nambu-Goto action becomes insufficient when we attempt to describe the excitation energy spectrum. Now there are additional contributions that can lead to large deformations even when the underlying displacements remain tiny. For example, a small torsional rotation around the axis of a long string can cause distant cross-sections to rotate through large angles. Similarly, if a long string is slightly bent its ends can move over considerable distances.

In the present Letter we propose an effective field theory approach to describe the excitation energy spectrum of a confining string. The string appears as a soliton in the effective field theory, which enables us to account for contributions such as bending and twisting at the (semi)classical level. This provides us with an appropriate background, to systematically investigate quantum corrections. Quantitatively, our arguments are based on the classical action [2]

$$\int d^4x \left[(\partial_\mu \mathbf{n})^2 + \frac{1}{4e^2} (\mathbf{n} \cdot \partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})^2 \right] \quad (2)$$

The order parameter \mathbf{n} is a three-component unit vector $\mathbf{n} \cdot \mathbf{n} = 1$ and e^2 is a coupling constant that determines the length scale. This action is known to support closed knotted strings as solitons [3] - [4]. According to [5], it also represents a universality class that describes SU(2) Yang-Mills theory in the long wavelength limit. Indeed, (2) does emerge from the Yang-Mills theory in a derivative expansion [6]. Thus it becomes natural to use its string solutions to inspect properties of the confining flux in the Yang-Mills theory. However, ultimately we wish to employ universality to propose that our results remain valid even beyond the steps that lead to (2). For this we shall combine duality arguments with a familiar problem in the classical theory of elasticity, the evaluation of the free energy that describes elastic deformations of a thin rod [7]. At least to the extent our results parallel those in the classical theory of elasticity, we expect their validity to reach beyond the specific details of (2).

Classically there are exactly two intrinsically small deformations of an elastic rod that can be accompanied by large global displacements, twisting about its axis and bending [7]. When a straight elastic rod is twisted around its axis, each transverse section becomes rotated by some relative angle. We align the coordinates so that the axis of the rod coincides with the z -axis and denote by τ the (local) twist angle, defined as the angle of rotation per unit length of the rod around its axis. For a constant τ a rod with length L acquires a total twist $\psi = \tau L$ along its length. Combining (1) with the results in [7] we then expect that the free energy of a straight, twisted confining string has the form

$$E_0 + E_{twist} = \sigma \cdot L + \varepsilon + (2\pi^2 \mathcal{C} + \frac{c}{\mathcal{T}^2}) \cdot \frac{\mathcal{T}^2}{L} + \mathcal{O}\left(\frac{1}{L^2}\right) = \sigma |\mathcal{T}| \cdot \left(\lambda + \epsilon + \frac{a}{\lambda} \right) + \mathcal{O}\left(\frac{1}{L^2}\right) \quad (3)$$

Here $\mathcal{T} = \frac{1}{2\pi}\psi$ counts the number of twists around the axis, \mathcal{C} is a form-factor that characterizes the string, and the variable $\lambda = L/|\mathcal{T}|$ denotes length per twist. We note that (3) exhibits a $\lambda \rightarrow a/\lambda$ duality, reminiscent of the T-duality in (super)string theories. When the number of twists \mathcal{T} remains fixed the self-dual point $\lambda = \sqrt{a}$ minimizes the ensuing contribution which yields

$$E_{min}(Q) \approx 2\sqrt{a} \cdot \sigma|\mathcal{T}| + \varepsilon + \mathcal{O}\left(\frac{1}{L^2}\right) \quad (4)$$

In the field theory model (2) a straight twisted string is described by an axially symmetric configuration, in cylindrical coordinates

$$\mathbf{n}(r, \varphi, z) = \begin{pmatrix} \sin(k\varphi + \tau z) \sin \theta(r) \\ \cos(k\varphi + \tau z) \sin \theta(r) \\ \cos \theta(r) \end{pmatrix} \quad (5)$$

where k is an integer that counts the degeneracy. We substitute (5) into (2), and by defining $\rho = er$ a dimensionless variable, we find for the energy of a string with length L

$$\frac{E}{2\pi} = \left\{ \int_0^\infty \rho d\rho \left(\theta_\rho^2 + k^2 \cdot \left[\frac{1 + \theta_\rho^2}{\rho^2} \cdot \sin^2 \theta \right] \right) \right\} \cdot L + \frac{4\pi^2 \mathcal{H}^2}{k} \cdot \left\{ \int_0^\infty \rho d\rho (1 + \theta_\rho^2) \cdot \sin^2 \theta \right\} \cdot \frac{e^2}{L} \quad (6)$$

We draw attention to the similarity in the functional forms of (3) and (6) which can be used to relate the parameters σ and \mathcal{C} to integrals over $\theta(\rho)$. Here we have introduced the Hopf invariant \mathcal{H} , explicitly with $F = (\mathbf{n}, d\mathbf{n} \wedge d\mathbf{n}) = dA$

$$\mathcal{H} = \frac{1}{2\pi} \int d^3x F \wedge A = \frac{1}{2\pi} k \tau L = \frac{k}{2\pi} \psi$$

so that for a straight string the Hopf invariant reduces to the number of twists $\mathcal{H} = \mathcal{T}$ that a configuration with length L makes around the z -axis (including the multiplicity k).

The profile $\theta(\rho)$ that minimizes the energy (6) solves the pertinent Euler-Lagrange equation. Since this equation depends nontrivially on the various parameters in (6), we expect *a priori* that the solution also attains a nontrivial, nonlinear dependence on the parameters. In particular we expect that for an actual minimum energy configuration the functional form of (6) in L does not stand.

We have performed an exhaustive numerical investigation of the parameter dependence for the configuration $\theta(\rho)$ that minimizes the energy (6). The result is described by a spectral function $f(k^2, \lambda)$. It depends on the integer k^2 and the dimensionless combination $\lambda = (Le)/(2\pi|\mathcal{H}|)$ which measures length per Hopf invariant. Quite surprisingly, we find that the functional form (3) *persists*: We find that to a very high degree of accuracy the energy of an actual straight string solution is interpolated by a simple rational spectral function

$$\frac{E_{min}(k^2, \lambda)}{4\pi^2 e} = |\mathcal{H}| \cdot f(k^2, \lambda) \approx |\mathcal{H}| \cdot \left(a\lambda + b + \frac{c}{\lambda} \right) \quad (7)$$

where a, b, c are k^2 dependent numerical coefficients. In figure 1 we plot (7) for $k = 1$, together with several numerically computed values of the energy. The high degree of accuracy of the rational spectral function (7) over a wide range of values in λ and k^2 leads us to propose that this simple rational form might actually be exact for the straight string solutions of (2) (or at least to a slight perturbation of (2) within its universality class).

We note that (7) has a definite, manifest modular structure as it should. Namely, suppose we split a string into two so that the total energy remains intact. Obviously, the spectral form of the energy for both of the resulting strings should also coincide with (7). Since the Hopf invariant $|\mathcal{H}|$ is additive and λ denotes length per Hopf invariant, we conclude that (7) indeed does have the requisite property under splitting and joining.

The definite rational form (7) leads to additional observations. For this we first note that if $\lambda = \lambda_c$ minimizes (7), in parallel with (3) the spectral function in (7) exhibits a $\lambda \rightarrow \lambda_c^2/\lambda$ duality. In order to interpret this duality we consider a string with Dirichlet-type boundary conditions that clamp the ends which keeps \mathcal{H} intact. According

to (7) the string has a tendency to adjust its length λ towards $\lambda = \lambda_c$ *i.e.* towards the self-dual point of the spectral function that minimizes the energy: When $\lambda < \lambda_c$ the value of λ has a tendency to increase, but when $\lambda > \lambda_c$ there is a tendency for λ to decrease.

We recall the relation between Hopf invariant \mathcal{H} , twist \mathcal{T} and writhe \mathcal{W} [8]

$$\mathcal{H} = \mathcal{T} + \mathcal{W} \quad (8)$$

From this we conclude that for $\lambda \neq \lambda_c$ the string has a tendency to *supercoil*: When $\lambda < \lambda_c$ it becomes energetically favourable for the string to dilate. This tends to decrease twist \mathcal{T} , hence there is a tendency to increase writhe \mathcal{W} . Similarly, when $\lambda > \lambda_c$ it becomes favourable to contract which can lead to an increase in twist at the expense of writhe. As a consequence the $\lambda \rightarrow \lambda_c^2/\lambda$ duality of the rational spectral function becomes related to supercoiling. It can map a supercoiled configuration with an excess amount of twist to a (dual) supercoiled configuration with an excess amount of writhe but with an equal amount of energy: Since the Hopf invariant \mathcal{H} does not change when we exchange $\mathcal{T} \leftrightarrow \mathcal{W}$, we have a twist-writhe duality mapping that exchanges $\mathcal{T} \leftrightarrow \mathcal{W}$ and sends $\lambda \rightarrow \lambda_c^2/\lambda$ but leaves the spectral function intact. The energy (7) is invariant under this twist-writhe duality transformation, it determines a symmetry in our effective field theory [9].

The previous arguments are based on the inspection of a straight string. But a configuration with a nontrivial writhe can not be straight, it must be bent. In order to proceed it becomes imperative to inspect the effects of bending. For this we recall from classical theory of elasticity that the free energy for bending of a thin elastic rod, when uniformly bent over a length L , has the form $E = \kappa L/R^2$ with R the (mean) radius of curvature, and κ a form factor that characterizes the rod [7].

We consider a confining string which has been bent *uniformly*, with a constant radius of curvature R . We assume that there is a constant rate of twist along the string. The writhe is similarly supposed to be uniformly distributed along the string, with a constant rate of increment ω . The length $L(\omega)$ of the string then scales in proportion to ω according to $L(\omega) = L_0 + c \cdot \omega R$. Here L_0 is the (planar) component of the length that does not contribute to the writhe, and c is a numerical constant; for simplicity we set $c = 1$. When we combine the free energy for bending with the ensuing free energy (3) for stretching and twisting, we conclude that the free energy of our string has the following ω dependence,

$$E(\omega) = \sigma L(\omega) + \alpha^2 \cdot \frac{(\mathcal{H} - \omega)^2}{L(\omega)} + \beta^2 \cdot \frac{L(\omega)}{R^2} \quad (9)$$

with α and β some parameters.

We first consider the minimum value of the free energy (9) when we minimize it *w.r.t.* the rate of increment ω , by keeping the Hopf invariant \mathcal{H} fixed. We get

$$E_{min} = 2(\rho - \alpha) \cdot \frac{\alpha}{R} \cdot \left(|\mathcal{H}| + \frac{L_0}{R} \right) \quad (10)$$

where we have defined $\rho^2 = \alpha^2 + \beta^2 + \sigma R^2$. We observe that (10) has the same functional form as (4). This suggests in particular, that the underlying dual structure persists. For this we subject the free energy to a minimization of ω , with the condition that both the Hopf invariant \mathcal{H} and the total length of the string $L = L_0 + \omega R$ remain fixed. We find

$$E_{min} = |\mathcal{H}| \cdot \frac{L}{S} \cdot \left(\sigma \cdot \frac{S}{|\mathcal{H}|} + (\alpha\beta)^2 \cdot \frac{|\mathcal{H}|}{S} \right) \quad (11)$$

where $S^2 = \alpha^2(L - L_0)^2 + \beta^2 L^2$. The dual structure is now manifest. We also note that both (10) and (11) are consistent with the expected modular property of a uniformly writhed and twisted string under splitting and joining.

In the field theory model (2) a stable, bent and twisted confining string is (locally) described by a closed soliton in the form of a torus knot [3], [4]. The radius of curvature of an actual soliton in general fails to be constant, as there are deviations due to interactions between the strands. But at least for small values of \mathcal{H} the deviations from an average R appear to remain small [4]. Consequently the main features of (9) should persist. Since an arbitrary space curve is described at each point by its torsion and radius of curvature, this suggests that the energy of a closed knotted

soliton should be qualitatively described in terms of modular invariant quantities only. This leads us to expect that the manifestly twist-writhe dual functional form of the spectral representation (7), (11) continues to hold, with the various parameters now integrals of \mathbf{n} evaluated for the soliton. *A priori* such parameters depend on the details of the soliton profile, but according to (7) this dependence appears to be quite weak. Since the soliton minimizes the energy it corresponds to the self-dual minimum energy point of the relevant spectral function, and according to (7), (10) this means that the energy spectrum of a closed knotted soliton admits the now-familiar functional form

$$E_{\text{soliton}} = \gamma \cdot (|\mathcal{H}| + \varepsilon) \quad (12)$$

with some parameter γ [10] In particular, since $|\mathcal{H}| = 1, 2, 3, \dots$ is an integer the energy spectrum will be equipartitioned, $E_n = \gamma \cdot (n + \varepsilon)$.

Due to the high complexity of the Euler-Lagrange equations of (2) it becomes difficult to actually check the accuracy of (12), or to inspect the underlying dual structure. Numerical simulations are very demanding [4], and at the moment we lack computational resources to perform exhaustive simulations. Thus we rely on the numerical results that have been recently published in [4]. There, the energy of various knotted solitons is evaluated for $Q_H = 1, \dots, 8$. In figure 2 we compare the data in [4] to the predictions of (12). When we account for the numerical uncertainties in the data as described in [4], we conclude that the agreement appears quite satisfactory. In particular, this suggests that the dual twist-writhe structure of (2) is indeed inherent.

In conclusion, we have inspected properties of a confining string by describing it as a soliton in an effective field theory. In particular, we have numerically studied the excitation energy spectrum of a straight string. We find that it can be described by a spectral function that admits a simple rational form, manifestly invariant under a duality transformation that relates twist and writhe. We have verified that the presently available 3D data [4] is also consistent with the simple rational form of the spectral function. This suggests that (2) is at least in the same universality class with a Lagrangian for which a manifestly dual, simple rational form of the spectral function is *exact*. It becomes natural to expect that our rational realization of the twist-writhe duality is distinct to an effective Lagrangian that accurately describes the properties of the confining flux in the Yang-Mills theory.

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 - [9] We note that for large enough deviations $\lambda - \lambda_c$ the rational form (7) must eventually fail: A line string becomes unstable against splitting into closed configurations with nontrivial \mathcal{H} .
 - [10] According to (11) γ and ε can depend on R *etc.* properties of the soliton. This may introduce implicit \mathcal{H} dependence, restricted by the lower bound in A.F. Vakulenko and L.V. Kapitanski, Dokl. Akad. Nauk USSR *248* (1979) 810

Figure Caption

Figure 1: A comparison of the rational function (7) with the energy values of the line solitons in (2), (6). The interpolation is in the sense of least square, which yields $a \approx 4.15...$, $b \approx 5.26...$ and $c \approx 1.42...$. The agreement is consistent with finite lattice size errors.

Figure 2: Comparison of the linear trajectory (12) (see also [10]) to numerics in [4]. Line/data with $+$ denotes the toroidal $\mathcal{H} = 1, \dots, 8$ solutions with slope/intercept $364/116$ and line/data with \bullet the minimal energy $\mathcal{H} = 1, \dots, 8$ solutions with slope/intercept $277/306$. (We note that [4] considered only interpolation with a vanishing intercept.)

figure 1

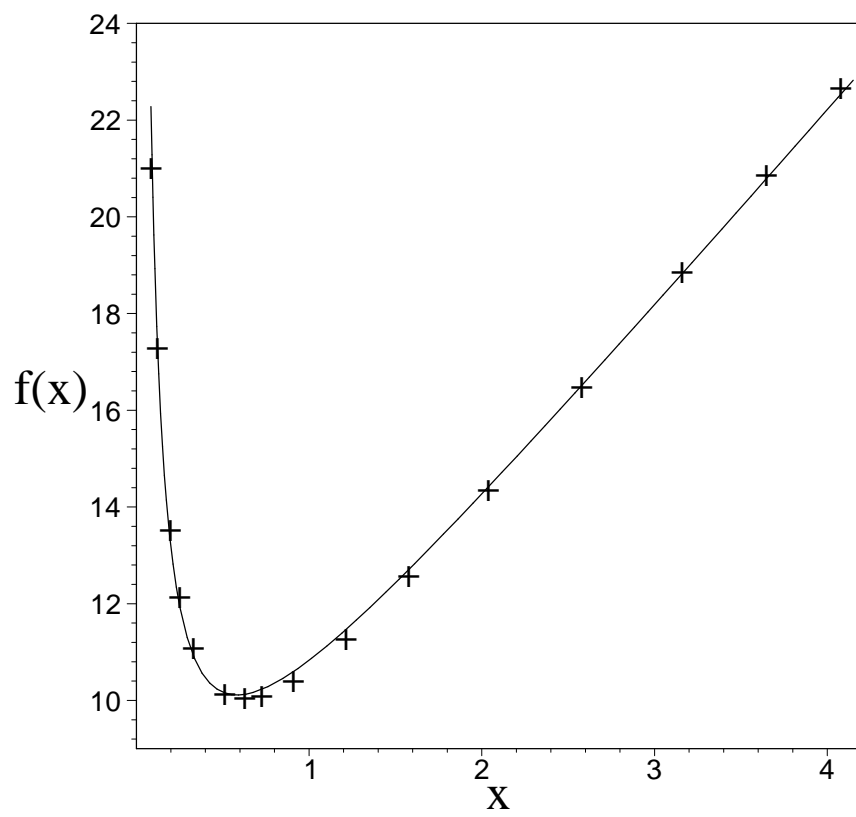


figure 2

